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The self-propulsion of a deformable body in a perfect fluid

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It is shown that a deformable body can move persistently from rest through a perfect fluid without having to produce vorticity in the fluid.

Introduction

It is well known that there is no force on a body in a uniform stream of perfect fluid (D'Alembert's paradox). However, the related question of whether a body that is initially at rest relative to the fluid can by deformation of its surface give itself a persistent velocity does not seem to have been considered.† In simpler language, we ask the question: can a fish swim in a perfect fluid? Of course, if circulation is allowed to be created in the fluid, by boundary-layer separation or the imposition of a Kutta condition at a sharp trailing edge, then momentum can be transferred between the fluid and the body, and the body can propel itself through the fluid. Taylor (1952), Lighthill (1960), Wu (1961) and others have studied the swimming of bodies in fluids of small viscosity. But if the viscosity is identically zero (as in a superfluid) so that the fluid motion is at all times irrotational with a single-valued velocity potential, the propulsive mechanisms associated with the appearance of vorticity in the fluid are not available. It might then be thought that the body would be unable to swim, or more precisely that it would be unable to move itself relative to the fluid by an arbitrary amount and have the same shape and structure at the end of the motion as at the beginning. The purpose of this note is to point out that this conclusion is false. The result will be demonstrated by constructing several fairly simple examples, but it will become apparent that any deformation which violates some fairly weak symmetry conditions will produce motion with a persistent component. Of course, a body can move itself vertically in a gravitational field by contracting or expanding so that its density is not equal to the fluid density, but we exclude such hydrostatically produced motions.

Motion in a straight line without rotation

The simplest case to consider is that of motion in a straight line in the absence of external forces for a body which is at all times symmetrical about the direction of motion. We suppose that the geometric centroid C of the body moves with

† With the exception of a recent paper by Benjamin & Ellis (1966), where it is pointed out that skew deformations of a body will tend to self-propel it through a fluid.

velocity W and that the centre of mass M moves with velocity $W+U$. For a heterogeneous body, the relative velocity of C and M is an arbitrary function of time determined by the internal structure; but $U \equiv 0$ if the body is homogeneous since C and M then coincide.

We suppose the motion is started from rest. Then the conservation of momentum states that at time t

$$M(W+U)+I = 0, \tag{1}$$

where M also denotes the mass of the body and I is the component of the fluid 'impulse' (Lamb 1932, Ch. VI) in the direction of motion. The fluid impulse can be interpreted as the linear momentum of the fluid (Landau & Lifshitz 1959, §11), but there are well-known difficulties as the volume integral of the velocity is only conditionally convergent, and the impulse is properly the integral with respect to time of the net pressure force acting over the body surface.

For irrotational motion with velocity potential ϕ (the convention is employed that $\mathbf{u} = \nabla\phi$), there is a result due to Kelvin (see Lamb 1932)

$$I = -\rho\int_B \phi\mathbf{n}.\mathbf{e}\,dS, \tag{2}$$

where $\mathbf{n}$ is the normal from the body into the fluid, $\mathbf{e}$ is a unit vector in the direction of motion, ρ is the fluid density, and the integral is over the body surface.

The velocity potential can be broken up into two parts,

$$\phi = \phi_T+\phi_D, \tag{3}$$

where ϕ_T is the 'translation potential' due to the motion of an instantaneously identical rigid body moving with velocity W, and ϕ_D is the 'deformation potential' due to the change in shape relative to the rigid body. The component of impulse can be broken up into contributions from the two parts of the velocity potential, i.e.

$$I = I_T+I_D. \tag{4}$$

Because

$$W = \frac{1}{V}\int_B (\mathbf{r}.\mathbf{e})(\mathbf{u}.\mathbf{n})\,dS, \tag{5}$$

where V is the volume of the body, and $\mathbf{r}$ is measured relative to the centroid, it can be deduced that the impulse can be expressed as integrals over a *sphere* at infinity as follows:

$$I_D = -3\rho\int_\infty \phi_D\mathbf{n}.\mathbf{e}\,dS; \tag{6}$$

$$\begin{aligned} I_T &= -3\rho\int_\infty \phi_T(\mathbf{n}.\mathbf{e})\,dS-\rho VW \\ &= mW, \quad \text{say,} \end{aligned} \tag{7}$$

where $m\,(>0)$ is called the virtual or apparent mass of the body. The formulae (6) and (7) hold without restriction on the degree of connectivity of the body. The only restriction is that the body does not swallow fluid, as the momentum of fluid enclosed by the body is not included in (6) and (7).

The equation of momentum conservation can then be written

$$[M+m(t)]W = -MU(t)-I_D(t), \tag{8}$$

where the quantities for which the time dependence is shown explicitly are functions only of the shape, structure and rate of deformation of the deformable body and are independent of W. It is clear that an arbitrary displacement can be effected without a permanent or net deformation of the body if m, U and I_D can be made to vary periodically with t in such a way that W has a non-zero time average, i.e.

$$\int_0^t W(t')\,dt' \to \overline{W}t \quad \text{as} \quad t \to \infty \tag{9}$$

with $\overline{W} \neq 0$. We shall now describe two different ways in which this can be accomplished, one for a heterogeneous and the other for a homogeneous body.

The case of a heterogeneous body

For a heterogeneous body, we can have $U \neq 0$, and it is simplest to suppose that the surface deformation has fore and aft symmetry so that $I_D \equiv 0$. Then $\overline{W}$ is positive if $U(t)$ and $m(t)-m(0)$ oscillate periodically in phase or with an in-phase component. For example, the body could be a light ellipsoidal shell of variable eccentricity but constant volume containing a heavy mass that can pull itself along the axis of revolution. The virtual mass is a function of the eccentricity and can be varied independently of the motion of the mass relative to the shell. If, for the sake of example,

$$U = U_0 \sin \omega t, \quad m = m_0(1+\alpha \sin \omega t), \tag{10}$$

it follows that

$$\overline{W} = \frac{2\alpha m_0 M}{(m_0+M)^2 - \alpha^2 m_0^2 + (m_0+M)\,[(m_0+M)^2 - m_0^2\alpha^2]^{\frac{1}{2}}}. \tag{11}$$

Alternatively, we could suppose that the shell was a sphere with pulsating radius $R(t)$. The virtual mass is $\frac{2}{3}\pi\rho R^3$. There is a net forward motion if R is less than its mean value when U is negative, and vice versa.

The physical explanation of the propulsion mechanism is clear. There is no hydrodynamic force on a body moving with uniform velocity, but there is one on an accelerating body, which is described by the virtual mass. Now if the centre of mass is moved backwards, the recoil will send the shell forward. If then the resistance or virtual mass is less when the shell goes forward than it is when the reverse recoil is moving the shell backwards, the distance covered during the forward motion exceeds that covered during the backwards motion and there is a net forward displacement during each cycle. If the deformation of the body stops, then the body comes immediately to rest (after being displaced by an arbitrary amount depending on the duration) so that a uniform motion cannot be produced. Note that there is no continuing transfer of momentum between the body and the fluid; the momentum of the body oscillates about a non-zero mean while the oscillating deformation continues. The situation is therefore very different from the inviscid propulsion mechanisms of Lighthill and Wu where there is a persistent transfer of momentum from body to fluid associated with the creation of vorticity. Also there is no energy dissipation in the present case and no net work is expended. There is of course a transfer of energy between body and fluid, but this is loss-free and reversible.

It is worthwhile considering briefly the effect of buoyancy forces. Equation (8) should then be replaced by a vector equation with an extra term on the right-hand side equal to the impulse of the buoyancy force during the motion. The problem is now more complicated as there is in general a buoyancy couple, even if the buoyancy force is zero, since the resultant buoyancy forces act through the geometrical centroid (in a uniform gravitational field) which does not coincide with the centre of mass, and the body will tend to rotate and set up rotation in the fluid. If the rotation is negligible, as will be the case for vertical motion or if the moment of inertia (either real or virtual) is large, then it is obvious that the velocity produced by the deformation is independent of the buoyancy forces. However, if the buoyancy couple is not negligible, or for that matter if the body is not moving parallel to an axis of symmetry so that there is a couple produced by the motion, the equations of motion become quite complicated and some care is required. It can be shown that motion in a given direction can be achieved by a rotation of the internal structure to compensate for the buoyancy couple or the couple produced by the motion itself, but we shall not give details here. In general, a body of zero buoyancy will move along a zig-zag path when the motion is produced by deformation and oscillation of the centre of mass.

Motion of a homogeneous body

For a homogeneous body, the centroid and centre of mass coincide and $U \equiv 0$. Motion can then be produced by asymmetrical deformations with non-zero values of I_D. Of course, m and I_D are not independent because they are both functionals of the shape, but we shall again show by constructing an example that persistent motion is possible. In fact, we shall demonstrate the stronger result that periodic deformations can give rise to a periodic I_D with non-zero mean. Unfortunately, a search for simple exact solutions to demonstrate these properties has proved unsuccessful, and the simplest three-dimensional analysis appears to be for a slightly deformable sphere.

Let us suppose the surface of the body has the equation in spherical polar coordinates ($\mu = \cos\theta$) with origin at the centre of mass and the polar axis in the direction of motion,

$$r = a + \epsilon_2(t)P_2(\mu) + \epsilon_3(t)P_3(\mu) + \eta_0(t) + \eta_1(t)P_1(\mu) + O(\epsilon^3). \tag{12}$$

Here ϵ_2 and ϵ_3 are small quantities of the first order of characteristic value ϵ, and η_0 and η_1 are of order ϵ^2 and are introduced so that the volume is constant and the centre of mass is at the origin to order ϵ^2. It is a straightforward calculation to show that

$$a\eta_0 = -\tfrac{2}{5}\epsilon_2^2 - \tfrac{2}{7}\epsilon_3^2, \quad a\eta_1 = -\tfrac{27}{35}\epsilon_2\epsilon_3. \tag{13}$$

It is necessary to work to second order, because from (6) it is apparent that a deformation which leaves the centre of mass at the origin gives no effect to first order. The calculation of the velocity potential to second order is straightforward but tedious. The details are of no interest and we just give the results:

$$m(t) = \tfrac{2}{3}\pi\rho a^3[1 - \tfrac{9}{5}(\epsilon_2/a)] + O(\epsilon^2), \tag{14}$$

$$I_D(t) = \tfrac{6}{7}\pi\rho a^2(2\dot{\epsilon}_2\epsilon_3 - 2\epsilon_2\dot{\epsilon}_3) + O(\epsilon^3). \tag{15†}$$

† The qualitative form of this result is also given by Benjamin & Ellis (1966).

It follows from (8) that for a body with density equal to the fluid density

$$W = (3/7a)(3\epsilon_2\dot{\epsilon}_3 - 2\dot{\epsilon}_2\epsilon_3) + O(\epsilon^3). \tag{16}$$

If therefore ϵ_2 and ϵ_3 have a component out of phase, $\overline{W} \neq 0$ and there is persistent motion. It is clear from (15) that $\bar{I}_D \neq 0$, and the motion is due to the impulse of the fluid due to the deformation having a non-zero mean. In this example, the deformation has two components, one with fore and aft symmetry and the other without. For motion to take place, the two components are out of phase, so that the shape of the body when the recoil (which is due entirely to the asymmetrical component) is sending it forward is different from the shape when the recoil is sending it back. Thus in essence the physical mechanism is the same as for the case of a heterogeneous body, and the general remarks made at the end of the previous section apply equally well here, except that buoyancy forces now uncouple completely. Thus we have demonstrated that a fish can swim in a perfect fluid.†

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† It is perhaps worth mentioning the obvious fact that if the body can swallow fluid, motion can also be produced by taking fluid in at the front and leaving it behind at the rear.